Vinti Solution for Free-Flight Rocket Trajectories

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In order to predict accurately the free-flight motion of a rocket near the earth (or other oblate body) for purposes of trajectory design and guidance, the gravitational effect of the equatorial bulge must be accounted for. This can be done by solving equations of motion which are based on a gravitational potential model developed by J. P. Vinti. In this paper, Vinti's analysis, originally intended for satellite orbits of many revolutions, is modified considerably in order to obtain a solution especially suited to the free-flight segments of ascent to satellite orbit, satellite deorbit, ballistic missile, and spacecraft skip re-entry trajectories. An important feature of the Vinti solution is that the trajectory calculations take only 0.05 sec on a digital computer—more than an order of magnitude faster than a numerical integration solution of the same precision.

Introduction

TO predict accurately the free-flight motion of a rocket, it is necessary to solve equations of motion which are based on an accurate mathematical model of the gravitational potential of the earth (or other central body). A first approximation to the solution can be obtained by using a simplified model for the earth—a point mass, for instance. The solution to the resultant, simplified equations of motion is called an "intermediary solution." In the case of the point mass approximation, the intermediary solution satisfies Kepler's laws.

To obtain a more accurate solution, the intermediary solution is modified by some sort of perturbation analysis to account for the terms that were omitted in simplifying the equations of motion. These higher order terms account for the disturbing effects due to the nonsphericity of the earth. Unfortunately, the equations that result from the perturbation analysis are often very unwieldy. It would be preferable if the relatively compact equations used for the intermediary orbit solution were sufficiently accurate that no perturbation analysis is needed. For near-earth orbits of less than about one revolution, this is the case for the intermediary orbit theory developed by J. P. Vinti.[1, 2]

When Vinti's intermediary solution is applied to the free flight segment of an earth-to-satellite, satellite-to-earth, or ballistic missile trajectory, the theoretical position error generally is less than 100 ft. This is due to the fact that the gravitational potential model used by Vinti is very close to the most accurate models derived from field observations.

The principal feature of the Vinti potential is that it permits the separation of the Hamilton-Jacobi equation in oblate spheriodal coordinates. The resultant intermediary orbit equations involve incomplete elliptic integrals that Vinti expands in series, some of which converge with the square root of the earth's oblateness parameter. If the independent variable is time or altitude, the orbit equations must be solved iteratively to obtain the position and velocity of the rocket. The iteration can be simplified greatly or even avoided in some cases if, instead of using Vinti's expansions, the solution is left in terms of canonical elliptic integrals. The simplification is partly due to the fact that, by introducing the amplitude elliptic function, one of the orbit equations can be inverted.

The elliptic integrals and functions can be evaluated rapidly from standard series that converge with the first power of the earth's oblateness parameter.

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Vinti Potential

The Vinti potential has the form

$$U = -k\rho(\rho^2 + c^2\eta^2)^{-1}$$

where k and c are constants and the oblate spheroidal coordinates are related to radial distance R and declination δ by

$$\rho^2 = R^2 - (1 - \eta^2)c^2 \qquad \rho\eta = R\sin\delta$$

The spheroidal azimuth coordinate is identical with the right ascension α. The origin of the spheroidal coordinate system is at the center of mass of the earth which is assumed to have rotational symmetry about the polar axis. The relationship between the oblate and spherical coordinates is illustrated in Fig. 1. A vertical cross section through the concentric surfaces of constant ρ (ellipsoids of revolution) and the orthogonal surfaces of constant η (hyperboloids of revolution) is shown. In terms of the distances r_1 and r_2 from the two foci, $\rho^2 = (r_1 + r_2)^2/4 - c^2$ and $\eta^2 = 1 - (r_1 - r_2)^2(2c)^{-2}$. c is the distance from either focus to the center of mass.

If the Vinti potential is expanded in spherical harmonics, the result is

$$U = -kR^{-1}[1 - c^2R^{-2}P_2(\sin\delta) + Rc^{4-4}P_4(\sin\delta) - \ldots]$$

where P_n are the Legendre polynomials. If k is chosen to be the earth's gravitational constant and $c^2 = 2R_e^2J/3$ where R_e is the equatorial radius and J is Jeffreys' oblateness parameter, then this potential represents the zeroth and second harmonics of the earth's potential exactly. The coefficient of the fourth harmonic of the Vinti potential is then $0.444J^2$. The coefficient of the fourth harmonic for the earth is $8D/35 = 0.91J^2$ according to Jeffreys' J/D ratio.[4] More recent estimates, based partly on satellite observations, give a smaller value for this ratio. Certainly at least half of the amplitude of the fourth harmonic is represented by the Vinti potential.

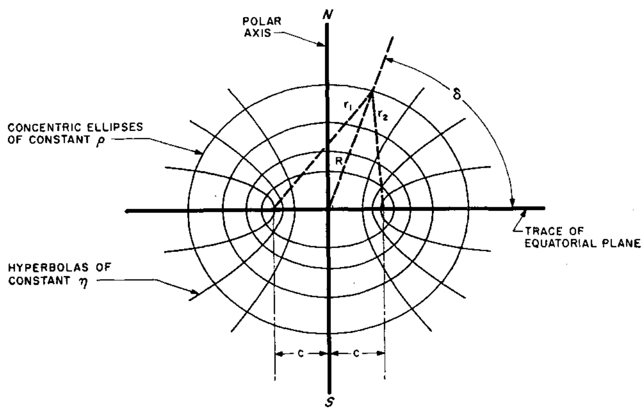

Fig. 1 Oblate spheroidal coordinates

Hamilton-Jacobi Method of Solution

The solution for the coordinates of the satellite (or rocket) motion by the Hamilton-Jacobi method is straightforward. By the standard procedure, the Hamilton-Jacobi equation is solved for Hamilton's characteristic function W in terms of the three constants of separation c_1, c_2, and c_3. c_1 can be identified with the total energy and c_3 with the polar component of angular momentum. c_2 reduces to the total angular momentum when $c^2 = 0$.

The equations of motion, in terms of the characteristic function, become

$$t + \beta_1 = \partial W/\partial c_1 \qquad \beta_2 = \partial W/\partial c_2 \qquad \beta_3 = \partial W/\partial c_3$$

where the three β constants are to be found from initial conditions and, except for a constant,

$$W = c_3\varphi \pm \int_{\rho_1}^{\rho} (\rho^2 + c^2)^{-1}[c^2c_3^2 + (-c_2^2 + 2k\rho + 2c_1\rho^2)(\rho^2 + c^2)]^{1/2}\, d\rho \pm \int_0^{\eta} (1 - \eta^2)^{-1}[-c_3^2 + (c_2^2 + 2c_1c^2\eta^2)(1 - \eta^2)]^{1/2}\, d\eta$$

The $\pm$ sign is to be taken according to whether the variable in the upper limit of the integrals is increasing or decreasing.

When the partial derivatives are expanded, the right-hand sides of the forementioned equations of motion will contain incomplete elliptic integrals. The details of the development will not be repeated here since they are covered in the comprehensive treatment by Vinti. To facilitate cross reference, Vinti's analysis and notation are followed as much as possible in the present section and in the Factorization Section.

Factorization

In order to reduce the elliptic integrals to canonical forms, it is necessary to factor the polynomials appearing under the square root of the forementioned integrands for W. The polynomial under the root in the η integrand factors immediately to

$$c_4^2[1 - (\eta/\eta_a)^2][1 - \kappa^2(\eta/\eta_a)^2]$$

where the constants are related to c_1, c_2, c_3, and c^2 as shown in the Calculations Section and

$$1 \geq \eta_a^2 > \eta^2 \qquad \kappa^2 \sim 10^{-3}(R_e/s)^2$$

Here R_e is the earth's equatorial radius and s the semilatus rectum of the orbit.

The polynomial under the root in the ρ integrand must be factored in the form

$$-2c_1(\rho_2 - \rho)(\rho - \rho_1)(\rho^2 + A\rho + B)$$

where

$$A = \rho_1 + \rho_2 + k/c_1 \quad \text{and} \quad B = -c^2(c_2^2 - c_3^2)(2c_1\rho_1\rho_2)^{-1}$$

By a method of successive approximations, the roots ρ_1 and ρ_2 may be found. Expressions for these roots, accurate to the second order in an oblateness parameter $k_0 \sim (R_e/s)^2\, 10^{-3}$, are given in the Calculations Section. For arbitrary accuracy, it is necessary to factor numerically. If errors up to about 300 ft are acceptable the second-order approximations can be used for trajectories of less than one orbit.

Solution in Terms of Canonical Elliptic Integrals

The elegant part of Vinti's analysis is in his expansion of the elliptic integrals in such a manner that he obtains complete separation of secular and periodic terms. Instead of following that analysis, the equations of motion will be expressed explicitly in terms of canonical elliptic integrals that then can be evaluated from standard power series expansions. These are power series in the square of the modulus of the elliptic functions, whereas some of Vinti's expansions are power series in the modulus and consequently converge more slowly. This modulus is proportional to the square root of the oblateness parameter J.

Incidentally, these standard power series can be separated into secular and periodic parts almost by inspection. However the resultant forms are more complicated than Vinti's, since complex parameters are involved.

The canonical elliptic integrals are: $F(\theta,\kappa)$, the incomplete elliptic integral of the first kind; $E(\theta,\kappa)$, the incomplete elliptic integral of the second kind; and $\Pi(\theta,\eta_a^2,\kappa)$, the incomplete elliptic integral of the third kind with parameter η_a^2. Their integral representations will not be repeated here, since they are well known.[3]

The elliptic integrals obtained by differentiating the characteristic function W can be reduced to combinations of these canonical forms. This can be done most easily by using the excellent handbook† of Byrd and Friedman.[3] The result is that the orbit equations can be put in the form

$$t + \beta_1 = (-2c_1ab)^{-1/2}\, S(f,\theta)$$

$$\beta_2/c_2 = -(2c_1ab)^{-1/2}\, F(f,\kappa_1) + \eta_a\, c_4^{-1}\, F(\theta,\kappa)$$

$$\beta_3 = \alpha - T(f,\theta)$$

where f and θ are related to ρ and η by

$$\tan\frac{f}{2} = \pm\left[\frac{\epsilon + 1 - (\Delta + s)/(\Delta + \rho)}{\epsilon - 1 + (\Delta + s)/(\Delta + \rho)}\right]^{1/2}$$

$$\eta = \eta_a \sin\theta$$

The quantities f, s, and ϵ are analogous to the true anomaly, semilatus rectum, and eccentricity of elliptic motion. Δ is a small quantity of the order sk_0.

The abbreviations $S(\ )$ and $T(\ )$ are defined by

$$S(f,\theta) = \Big\{[ll_2 - 2\Delta(\Delta + s)]\, R(f,\epsilon,\kappa_1) + (\Delta^2 + ll_1)\, F(f,\kappa_1) + l\epsilon^2\left[F(f,\kappa_1) - E(f,\kappa_1) + \frac{\epsilon(\sin f)(1 - \kappa_1^2\sin^2 f)^{1/2}}{1 + \epsilon\cos f}\right] + l_4[F(\theta,\kappa) - E(\theta,\kappa)]\Big\}$$

$$T(f,\theta) = c_3[\eta_a c_4^{-1}\Pi(\theta,\eta_a^2,\kappa) - (-2c_1ab)^{-1/2}\, c^2\, P(f,\kappa_1)]$$

where

$$P(f,\kappa_1) = (\Delta^2 + c^2)^{-1}[F(f,\kappa_1) + (\Delta + s)\, Q(f,\kappa_1)]$$

$$Q(f,\kappa_1) = \int_0^{F(f,\kappa_1)} \frac{(s - \Delta - 2\Delta\epsilon\, cnv)\, dv}{(s - \Delta\epsilon\, cnv)^2 + c^2(1 + \epsilon\, cnv)^2}$$

$$R(f,\epsilon,\kappa_1) = (1 - \epsilon^2)^{-1}\Pi[f,\epsilon^2/(\epsilon^2 - 1),\kappa_1] - \epsilon l_3 \tan^{-1}[l_3^{-1}(1 - \kappa_1^2\sin^2 f)^{-1/2}\sin f]$$

and Q is handled in the Evaluation Section. The arctangent is an angle in the first or fourth quadrants and is considered single valued. The elliptic integral moduli κ and κ_1 are of the order $(\kappa_0)^{1/2}$. All the constants appearing in these expressions are functions of the initial conditions and geophysical constants, as given in the Calculations Section. If the roots ρ_1 and ρ_2 had been found exactly, by numerical factoring, then the forementioned equations would give the exact solution for the Vinti potential.

† Note that there is a factor g missing from the right side of formula 259.04, used in connection with the Vinti solution.

When $c = 0$, the equations of motion reduce to the familiar forms

$$t + \beta_1 = (-2c_1\rho_2\rho_1)^{-1/2}(1-\epsilon^2)^{-3/2}s^2(E - \epsilon \sin E)$$
$$\beta_2 = -f + \theta$$
$$\beta_3 = \alpha - \tan^{-1}[(1-\eta_a^2)^{1/2}\tan\theta]$$

where

$$E = 2\tan^{-1}\{[(1-\epsilon)/(1+\epsilon)]^{1/2}\tan(f/2)\}$$

f is the true anomaly, ρ_2 is the apogee, ρ_1 the perigee, ϵ the eccentricity, s the semilatus rectum, and η_a the sine of the inclination of the orbital plane. It is apparent that β_2 becomes the argument of perigee and β_3 the longitude of the ascending node.

Terminal Point Coordinates

The preceding orbit equations of motion apply to a variety of rocket trajectories as well as to earth satellite orbits. For instance, the ascent trajectories for most satellite missions include free-flight, or "coast," segments between the power flight maneuvers. The descent trajectory for deorbit from a satellite orbit also contains a free-flight segment. The free flight segment of a ballistic missile trajectory and the skip portion of a spacecraft re-entry trajectory fall into this category.

If the initial position and velocity are given, the orbit equations can be used to find the terminal point of the free-flight trajectory segment. An iterative solution is required if the terminal point is specified in terms of elapsed time that is usually the independent variable in ascent trajectories. For descent trajectories, including ballistic missile re-entry, the free-flight trajectory usually is terminated at the "top" of the atmosphere. This can be given as a fixed altitude above the ellipsoidal earth, or, more approximately, by a surface of constant ρ. In the former case a simple iterative solution is needed, whereas no iteration is needed in the latter case. The surface of constant ρ is very close (within $\pm$ 1 mile) to a spherical surface.

If the elapsed time (coast time) t is given, the terminal values of ρ and η are found as follows. An initial f estimate, f^*, is obtained from the central force field equations given previously, for $c = 0$. A Newton-Raphson iteration procedure can be used to solve the time equation. Next the exact orbit equations involving t, f, and θ are solved iteratively for f, using f^* as the starting value. A Newton-Raphson procedure can be employed again, using the time derivative expression, df/dt, obtained for the $c = 0$ case. The remaining coordinates are found from

$$\eta = \eta_a \sin\theta \qquad \rho = [(\Delta + s)/(1 + \epsilon\cos f)] - \Delta$$

If the terminal point is given in terms of a surface of constant ρ, then the computational procedure shown in the Calculations Section is used to obtain η.

If the terminal altitude h above the earth's surface is given, the computational procedure given in the Calculations Section has to be altered. In the group of four equations for finding the terminal value of η, the first equation becomes simply $\rho = R_e$, as the initial approximation for ρ. After η is found, a better value for ρ is obtained from the relation between ρ and η on a surface of altitude h above the earth's ellipsoid (see the Appendix). Then f, θ, and η are recalculated using the improved value of ρ. This iterative procedure is repeated twice, using the new value of ρ for each cycle.

Evaluation of Canonical Forms

The power series expansions for the canonical forms, to the order of the fourth power of the modulus are

$$F(\theta,\kappa) = \theta + \kappa^2 t_2(\theta)/2 + 3\kappa^4(3\,t_2(\theta) - \sin^3\theta\cos\theta)/32$$

where

$$t_2(\theta) = (\theta - \sin\theta\cos\theta)/2$$
$$F(\theta,\kappa) - E(\theta,\kappa) = \kappa^2 t_2(\theta) + \kappa^4[3t_2(\theta) - \sin^3\theta\cos\theta]/8$$
$$\Pi(\theta,\eta_a^2,\kappa) = b_0 + \kappa^2(b_0 - \theta)\eta_a^{-2}/2 + \kappa^4(3\eta_a^2\sin\theta\cos\theta + 6b_0 - 6\theta - 3\eta_a^2\theta)\eta_a^{-4}/16$$

where

$$b_0 = (1-\eta_a^2)^{-1/2}\tan^{-1}[(1-\eta_a^2)^{1/2}\tan\theta]$$

and the arctangent is rendered single valued by requiring $b_0(1-\eta_a^2)^{1/2} = \theta$ whenever θ is a multiple of $\pi/2$. The amplitude elliptic function has the expansion

$$am(x,\kappa) = x/K + (\kappa^2/8)\sin(2x/K)$$

where

$$K = 1 + \kappa^2/4 + 9\kappa^4/64$$

The complete series expansions are given on pp. 299–303 of Ref. 3. Although this reference states that θ must be in the first quadrant, these expansions are actually valid for any real value of the argument θ. It might be added here that the term of $0(\kappa^4)$ was omitted intentionally from the expansion for the amplitude elliptic function due to the fact that the numerical coefficient of this term is itself $0(\kappa)$.

The integral $Q(f,\kappa_1)$ can be expressed in terms of two incomplete elliptic integrals of the third kind with complex parameters, so that it has the standard form. However, complex quantities can be avoided by expanding the integrand in powers of Δ and c^2 and employing term by term integration. The result, for $P(f,\kappa_1)$, is to order c^2/s^2

$$s^2P = (1 - c^2s^{-2})F(f,\kappa_1) + 2\epsilon(1 + \Delta s^{-1} - 2c^2s^{-2}) \times \sin f[1 - (\kappa_1^2/6)\sin^2 f] + \epsilon^2(1 + 4\Delta s^{-1} - 3c^2s^{-2})F(f,\kappa_1)$$

Calculations

The calculations to be carried out for the determination of the rocket's terminal coordinates in inertial space are summarized below. The inputs required are the initial position and velocity components in rectangular equatorial coordinates and initial time. The geophysical constants required are the earth's gravitational constant k, equatorial radius R_e, ellipticity e, and oblateness parameter J as defined by Jeffreys. The outputs are terminal time t, right ascension α, declination δ, and radial distance R from the earth's center. The Appendix gives equations for finding terminal velocity. The subscript i quantities refer to the initial point and the unsubscripted quantities to the terminal point.

The constants that appear in the orbit equations are calculated in the following order:

$$c^2 = 2R_e^2J/3$$
$$R_i^2 = X_i^2 + Y_i^2 + Z_i^2 \qquad \alpha_i = \tan^{-1}(Y_i/X_i)$$
$$\rho_i = \{(R_i^2 - c^2) + [(R_i^2 - c^2)^2 + 4c^2Z_i^2]^{1/2}\}^{1/2}2^{-1/2}$$
$$\eta_i = Z_i/\rho_i$$
$$c_1 = (\dot X_i^2 + \dot Y_i^2 + \dot Z_i^2)/2 - k\rho_i(\rho_i^2 + c^2\eta_i^2)^{-1}$$
$$c_3 = X_i\dot Y_i - Y_i\dot X_i$$
$$c_2^2 = \frac{[\rho_i\dot Z_i - \eta_i(X_i\dot X_i + Y_i\dot Y_i + Z_i\dot Z_i)]^2 + c_3^2}{(1-\eta_i^2)} - 2c_1c^2\eta_i^2$$
$$c_4 = (c_2^2 - c_3^2)^{1/2} \qquad c_5 = c_2^2 - 2c^2c_1$$
$$w = (1 + 8c^2c_1c_4^2c_5^{-2})^{1/2}$$
$$\kappa^2 = (1-w)/(1+w) \qquad \eta_a^2 = 2c_4^2(1+w)^{-1}c_5^{-1}$$

$$x^2 = -2c_1c_2^2k^{-2} \qquad y^2 = c_3^2c_2^{-2} \qquad k_0 = -2c_1c^2c_2^{-2}$$

$$I = 1 + k_0(x^2y^2 - 4y^2) - k_0^2y^2(12x^2 - x^4 - 20x^2y^2 - 16 + 32y^2 + x^4y^2) + \ldots$$

$$H = 1 - k_0x^2y^2 - k_0^2x^2y^2(2x^2 - 3x^2y^2 - 4 + 8y^2) + \ldots$$

$$\rho_1 = -k[H - (H^2 - x^2I)^{1/2}](2c_1)^{-1}$$

$$\rho_2 = -k[H + (H^2 - x^2I)^{1/2}](2c_1)^{-1}$$

$$B = -c^2c_4^2(2c_1\rho_1\rho_2)^{-1}$$

$$a = (2\rho_2^2 + \rho_1\rho_2 + \rho_2k/c_1 + B)^{1/2}$$

$$b = (2\rho_1^2 + \rho_1\rho_2 + \rho_1 k/c_1 + B)^{1/2}$$

$$\kappa_1^2 = [(\rho_2 - \rho_1)^2 - (a - b)^2](4ab)^{-1}$$

$$\epsilon = (a - b)(a + b)^{-1} \qquad s = (\rho_2b + \rho_1a)(a + b)^{-1}$$

$$\Delta = (\rho_2b - \rho_1a)(a - b)^{-1}$$

$$l = (s + \Delta)^2(\epsilon^2 - 1)^{-1} [\epsilon^2 + \kappa_1^2(1 - \epsilon^2)]^{-1}$$

$$l_1 = \kappa_1^2(1 - \epsilon^2) \qquad l_2 = -\epsilon^2 - 2\kappa_1^2(1 - \epsilon^2)$$

$$l_3 = (1 - \epsilon^2)^{1/2}[\epsilon^2 + \kappa_1^2(1 - \epsilon^2)]^{-1/2}$$

$$l_4 = c^2\eta_a^3 \kappa^{-2}c_4^{-1}(-2c_1ab)^{1/2}$$

$$f_i = \left\{2 \tan^{-1}\left[\frac{\epsilon + 1 - (\Delta + s)/(\Delta + \rho_i)}{\epsilon - 1 + (\Delta + s)/(\Delta + \rho_i)}\right]^{1/2} - \pi\right\} \times \operatorname{sgn}\gamma + \pi$$

where $\operatorname{sgn}\gamma = \operatorname{sgn}(X\dot{X} + Y\dot{Y} + Z\dot{Z})$

$$\theta_i = \tan^{-1}[\eta_i\eta_a^{-1}/(1 - \eta_i^2\eta_a^{-2})^{1/2}]$$

where the root has the same signature as

$$\eta_a^{-1}[\rho_i\dot{Z}_i - \eta_i(X_i\dot{X}_i + Y_i\dot{Y}_i + Z_i\dot{Z}_i)]$$

$$\beta_1 = (-2c_1ab)^{-1/2} S(f_i,\theta_i) - t_i$$

$$\beta_2/c_2 = -(-2c_1ab)^{-1/2} F(f_i,\kappa_1) + \eta_ac_4^{-1}F(\theta_i,\kappa)$$

$$\beta_3 = \alpha_i - T(f_i,\theta_i)$$

The terminal value of η on a surface of constant ρ (fitted to an altitude h at the equator) is found from

$$\rho = [(R_e + h)^2 - c^2]^{1/2}$$

$$= -2 \tan^{-1}\left[\frac{\epsilon + 1 - (\Delta + s)/(\Delta + \rho)}{\epsilon - 1 + (\Delta + s)/(\Delta + \rho)}\right]^{1/2} + 2\pi$$

$$\theta = am\left[\frac{\beta_2c_4}{\eta_ac_2} + c_4 \frac{(-2c_1ab)^{-1/2}}{\eta_a} F(f,\kappa_1),\kappa\right]$$

$$\eta = \eta_a \sin\theta$$

If the terminal point occurs before apogee, then the sign of the right side of the equation for f is reversed and the 2π term is dropped.

The orbit equations for finding the terminal time t and right ascension α are

$$t + \beta_1 = (-2c_1ab)^{-1/2} S(f,\theta) \qquad \alpha = \beta_3 + T(f,\theta)$$

Radial distance R and declination δ are found from

$$R = [\rho^2 + (1 - \eta^2)c^2]^{1/2} \qquad \sin\delta = \rho\eta/R$$

Remarks

Based on the preceding relations, a FAP digital computer program for calculating the terminal point of the trajectory requires less than 1000 storage locations, including elliptic integral subroutines, and takes 0.05 sec to execute, in terms of the IBM 7090 speed.

When the Vinti potential is used, the terminal position error, caused by the neglect of part of the fourth harmonic of the earth's potential, can be bounded quite simply. The residual acceleration due to the neglected part of the fourth harmonic is bounded by 4.1×10^{-5} ft/sec². For a free-flight trajectory of 2000-sec duration, an error bound of 62 ft is obtained by using the formula for a constant acceleration.

The terminal position obtained from the Vinti solution, for a 2000-sec trajectory, was compared with that obtained by precision numerical integration. The difference was only 23 ft.

The power series for the canonical forms were truncated at the fourth-order term in the modulus κ. It can be shown that the resultant error is less than $\kappa^6/6 \simeq 10^{-9} (R_e/s)^2/6$ rad where R_e is the earth's equatorial radius and s the semiparameter of the approximating trajectory ellipse. For most trajectories of practical interest, s is large enough that the truncation error will be less than the error caused by the uncertainty in the measured values of the geophysical constants.

Appendix

Additional quantities, such as position and velocity in equatorial coordinates, are needed frequently. For convenient reference, the necessary relations, derived from the expressions in the main text, are given here. The terminal position in inertial equatorial coordinates is given by

$$Z = \rho\eta \qquad X = (R^2 - Z^2)^{1/2} \cos\alpha \qquad Y = X \tan\alpha$$

The inertial velocity magnitude is

$$V = (2c_1 + 2k\rho(\rho^2 + c^2\eta^2)^{-1})^{1/2}$$

The velocity components in inertial equatorial coordinates are found from the sequence of computations

$$\dot{\theta} = [(1 - \eta^2)(c_2^2 + 2c^2 c_1\eta^2) - c_3^2]^{1/2} \times (\rho^2 + c^2\eta^2)^{-1} |(\eta_a \cos\theta)^{-1}|$$

$$\dot{f} = (-2c_1ab)^{1/2}(1 - \kappa^2 \sin^2\theta)^{-1/2}(1 - \kappa_1^2 \sin^2 f)^{1/2}\eta_a c_4^{-1}\dot{\theta}$$

$$\dot{\eta} = \dot{\theta}\eta_a \cos\theta \qquad \dot{\rho} = \dot{f}\epsilon(\rho + \Delta)^2(\sin f)(s + \Delta)^{-1}$$

$$R\dot{R} = \rho\dot{\rho} - c^2\eta\dot{\eta} \qquad \dot{Z} = \rho\dot{\eta} + \dot{\rho}\eta$$

$$\dot{X} = (R\dot{R} - X^{-1}Yc_3 - Z\dot{Z})(X + X^{-1}Y^2)^{-1}$$

$$\dot{Y} = X^{-1}(c_3 + Y\dot{X})$$

The flight path angle γ is found from

$$\sin\gamma = \dot{R}V^{-1}$$

The azimuth angle of the velocity vector is found from

$$\sin\beta = (X\dot{Y} - Y\dot{X})(V \cos\gamma)^{-1}(X^2 + Y^2)^{-1/2}$$

$$\cos\beta = (R\dot{Z} - ZV \sin\gamma)(\cos\delta)^{-1}$$

In oblate spheroidal coordinates, a surface of constant altitude h above the earth's ellipsoid that has ellipticity e and equatorial radius R_e is given by

$$\rho = h + \rho^*(\eta)\rho^{**}(\eta) + 0(e^3R_e) + 0(eh)$$

where

$$\rho^*(\eta) = R_e[1 - (e + \tfrac{3}{2}e^2)\eta^2 + \tfrac{3}{2}e^2\eta^4 + ec^2\eta^2(1 - \eta^2)R_e^{-2}]$$

$$\rho^{**}(\eta) = 1 - c^2(1 - \eta^2)R_e^{-2}/2 - c^4(1 - \eta^2)^2 R_e^{-4}/8 - c^2 e\eta^2(1 - \eta^2)R_e^{-2}$$

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